Research monograph # 64
April 2016

The Mathematical Territory Between Direct Modelling and Proficiency

By Dr. Alex Lawson, Lakehead University

Tips for Teaching Math Facts

1. Recognize that progressively learning math facts can lay important foundations for later mathematics.
2. Support the progression of increasingly sophisticated calculation strategies rather than jumping to memorization.
3. Pose well-constructed problems to guide this progression and encourage student-generated strategies.
4. Memorize for efficient calculations after students' reasoning strategies are well-established.

The potential learning that exists in the territory between direct modelling and memorization of facts is foundational for a great deal of later mathematics and for mental fluency. How should children come to know their math facts?

Over the past few years, there has been a media outcry over children's reputed lack of mastery of number facts and teachers' apparent disinterest in having students memorize them, as this excerpt from the *Guelph Mercury* indicates:

> The education bureaucracy has long held a disdain for teaching methods that emphasize "math facts," such as multiplication tables and simple arithmetic such 7 + 8 = 15 . . . . Instead, Ontario schools mostly use "discovery math," which allows students to explore concepts according to their own personal learning style.[1]

While such dichotomies make for good press, they misrepresent the situation. Most educators believe all children *should* learn number facts; however, they also believe that children *should* learn and understand much more math. Just memorizing the facts is no longer enough. In fact, it never was.

Past Practice

Many of us learned our facts by starting with direct modelling – usually with concrete objects.[2] To add 5 + 7, for example, we might have:

1. counted a first set of 5 objects
2. counted a second set of 7 objects
3. counted from 1 again to reach 12 (i.e., the strategy of *counting three times*)

Once we had mastered this method of addition, we would have memorized 5 + 7 = 12 as one isolated fact.

Unfortunately, this quick shift from direct modelling to memorization came at a cost for many students.[3,4] The potential learning that exists in the territory between direct modelling and memorization of facts is foundational for a great deal of later mathematics and for mental fluency.

What Math Lies Between Direct Modelling and Fact Memorization?

Students who, over time, construct a series of increasingly sophisticated personal strategies for solving calculations often learn mathematics that serves them in and beyond the classroom. In classrooms where teachers support the development of increasingly sophisticated addition and subtraction calculation strategies, children's methods reflect three general phases of progression before they become proficient.[2,5]

Phase 1: Direct Modelling and Counting. Consider our previous calculation, 5 + 7. During this phase, children use strategies such as counting three times (described above). From here, children begin to move away from fully modelling the problem.

Phase 2: Counting More Efficiently. Children use strategies such as *counting on* from 5, stating 6, 7, and so on, raising a finger to track each count of a mental number line until they reach seven fingers raised. In doing so, they shift from counting the direct (concrete) model of the second number, to continuing the mental number sequence in their mind from the first model. Their fingers are no longer physical objects to be counted but, instead, a mechanism for simultaneously "counting the count"[6] and tracking it.

Phase 3: Working With the Numbers. This phase illustrates the mathematical advantages of supporting children's evolving progression of calculation methods over quick memorization of facts, as children begin working with numbers rather than counting. Initially, they may use some form of a double.[7] For example, to determine 5 + 7, some children will break up the 7 in order to give 1 to the 5, transforming the expression into 6 + 6, a double they know. Mathematically, they think "5 + (1 + 6) = (5 + 1) + 6," capitalizing on the associative property of addition to make the numbers into something they know.[8] While children will not know the name of the property, they do know that breaking up the addends and re-associating them will result in the same sum, or that a + (b + c) = (a + b) + c. This begins the foundation of algebraic thinking.

With instructional support from their teacher, children may later – among a number of other increasingly sophisticated strategies – learn to decompose numbers in order to add using the strategy *up over ten*. With the example of 5 + 7, they may reverse the addends: To add 7 + 5, they may first add 3 (from the 5) to the 7 to get to 10 and then add the remaining 2 to get 12 (see Figure 1).

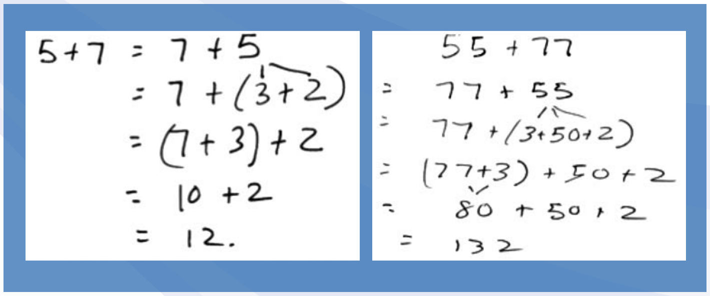

Figure 1: Up Over Ten Strategy for Single- and Double-Digit Addition.

When children reach these later strategies, like up over ten to solve 5 + 7 = 12 with speed and ease, some will commit the equation to memory as a fact, while others will need some short targeted drill.[9] Equally important, these children will have built foundations for later mathematics, such as algebraic generalizations (a + b = b + a) and strategic efficiencies (breaking apart the second addend to get to ten and then adding the remainder).

Later, these students can employ the same strategy with double-digit calculations. For example, when solving 55 + 77, they can commute the addends to start with 77; they can then break apart the 55 to add up by 3 to the nearest decade number (80), add 50, and add the remaining 2 to get 132 (see Figure 1).

Subtraction and the Roots of Algebra and Increasing Efficiency

It is with subtraction strategies that we see more clearly the foundations of future mathematics. Consider 14 – 8. Initially, as in addition, children will count three times by counting out 14 objects, removing 8, and recounting the remaining objects to find 6. As children shift to the next phase, they may eventually use the strategy of *counting on to subtract*. (Figure 2 illustrates both strategies).

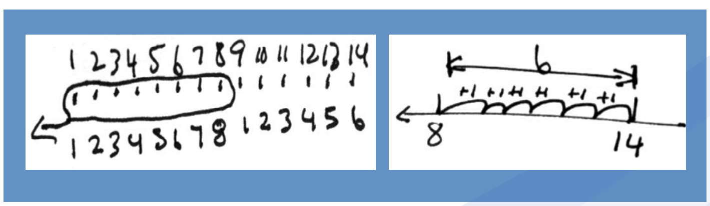

Figure 2: Subtraction as Removal (left) and Subtraction as Distance Between Numbers (right)

Rather than thinking of subtraction as simply removal, these children are now beginning to think about the distance between two numbers.[8] This is a profoundly different understanding of subtraction, and it is foundational to developing more efficient mental mathematics. Children who understand subtraction as the distance between numbers can solve subtraction calculations by adding up. They can also solve subtraction calculations by maintaining the distance (difference) while shifting numbers mentally to a nearby decade number. For example, the calculation 54 – 18 can be shifted up by 2 to the easier calculation of 56 – 20 (see Figure 3).

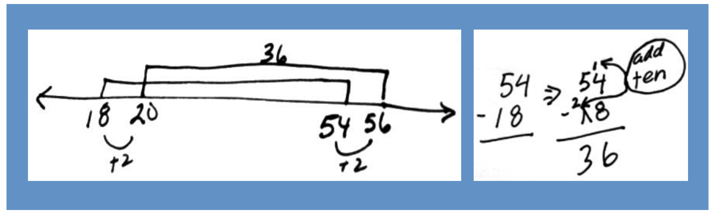

Figure 3: Solving 54 – 18 Using Constant Difference (left) and the Traditional British Subtraction Algorithm (right)

With further teacher support, students will be able to use this *constant difference* strategy to understand, for example, why the often-mysterious traditional British subtraction algorithm (which standardizes this strategy) works. They will be better poised to determine why adding ten to the minuend (top number) and adding ten to the subtrahend (bottom number) will give you the same difference, that is, 54 – 18 = (54 + 10) – (18 + 10).

These are but a few of the types of mathematical understandings and strategic efficiencies that children can achieve if they work through a range of increasingly efficient calculation methods rather than simply jumping to memorization. That said, another contention in the opening commentary must still be addressed.

Are Children Inventing Their Own Methods for Calculation?

The answer in effective classrooms is both yes and no. We have increasing evidence that teachers whose classes achieve the greatest amount of learning use a well-executed "guided-discovery" approach for learning facts,[10] rather than direct instruction (which most of us would have experienced) or discovery mathematics (which the *Guelph Mercury* editorialist discussed). In classrooms where a guided-discovery approach is effectively used, we may hear conversations such as the one in the following excerpt.

> ***Teacher:*** Your friend Natalie had 5 gummies and you gave her 7 more. How many did she have then?
> ***Student:*** 12.
> ***Teacher:*** Did you use the same strategy as last time?
> ***Student:*** Yes, I counted on.
> ***Teacher:*** You like that strategy?
> ***Student:*** Yes I invented it. My friends use it in class.[11]

The student believed she had invented counting on and noted with pride that her classmates used the strategy. This invention was, of course, guided – the teacher created a problem scenario where children would be more likely to develop and use this strategy (the first addend was hidden) and when the strategy came up, the teacher highlighted it and worked with children who were ready to use it. She knew the continuum of strategies from earliest to more sophisticated and she was pressing her students to move beyond counting three times. The notion that children develop their own calculation methods randomly is far from what actually happens in an effective mathematics classroom.

In Sum

Children should learn their number facts. However, they would benefit from learning these facts by using an increasingly sophisticated series of strategies rather than by jumping directly to memorization. As they work with these strategies, children develop deepening mathematical understandings that can be drawn on in everyday life as well as in junior- and high-school mathematics courses. Students who work through and become competent using increasingly sophisticated strategies do so, not through direct instruction, but rather as a result of teachers posing well-constructed problems that elicit and work with these evolving strategies, augmented by extensive practice in different contexts. Once children have worked through these reasoning strategies, they can memorize whatever facts have not yet become automatic using targeted drills.[9]

References

1. Wake-up call needed on math skills [Editorial]. (2014, September 3). Guelph Mercury, p. A8.
2. Fuson, K. (2003). Developing mathematical power in whole number operations. In J. Kilpatrick (Ed.), *A research companion to principles and standards for school mathematics* (pp. 68–94). Reston, VA: National Council of Teachers of Mathematics.
3. Henry, V., & Brown, R. (2008). First-grade basic facts: An investigation into teaching and learning of an accelerated, high-demand memorization standard. *Journal for Research in Mathematics Education, 39*, 153–183.
4. Baroody, A. (2006). Why children have difficulties mastering the basic number combinations and how to help them. *Teaching Children Mathematics, 13*, 22–31.
5. Lawson, A. (2015). *What to look for: Understanding and developing student thinking in early numeracy.* Toronto, ON: Pearson.
6. Sarama, J., & Clements, D. (2009). *Early childhood mathematics education research: Learning trajectories for young children.* New York, NY: Routledge.
7. Kamii, C. (2000). *Young children reinvent arithmetic: Implications of Piaget's Theory* (2nd ed.). New York, NY: Teachers College Press.
8. Fosnot, C. T., & Dolk, M. (2001). *Young mathematicians at work: Constructing numbers sense, addition, and subtraction.* Portsmouth: Heinemann.
9. van de Walle, J., Karp, K., Bay-Williams, J., McGarvey, L., & Folk, S. (2015). *Elementary and middle school mathematics: Teaching developmentally.* (4th Canadian ed.). Toronto: Pearson.
10. Gravemeijer, K., & van Galen, F. (2003). Making mathematics reasonable in school. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *A research companion for the principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.
11. Lawson, A. (2005) [Gr. 1 children's mathematical thinking]. Unpublished interview raw video data.

What Works? Research into Practice is produced by a partnership between the Ontario Association of Deans of Education and the Student Achievement Division.

To learn more about how to write a monograph **click here**
Mobilizing Research into Practice in Meaningful Ways
By Dr. Michelann Parr and Dr. Terry Campbell Co-editors

What Works? is updated and posted at: www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/WhatWorks.html

ISSN 1913-1097 What Works? Research Into Practice (Print)
ISSN 1913-1100 What Works? Research Into Practice (Online)